

On existence and uniqueness of positive solutions for integral boundary boundary value problems *

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Abstract: By applying the monotone iterative technique, we obtain the existence and uniqueness of $C^1[0, 1]$ positive solutions in some set for singular boundary value problems of second order ordinary differential equations with integral boundary conditions.

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1 Introduction and the main result

In this paper, we consider the existence of positive solutions for the following nonlinear singular boundary value problem:

$$\begin{cases} -u'' + k^2u = f(t, u), & t \in (0, 1), \\ u(0) = 0, \quad u(1) = \int_0^1 u(t) dA(t), \end{cases} \quad (1.1)$$

where A is right continuous on $[0, 1)$, left continuous at $t = 1$, and nondecreasing on $[0, 1)$, with $A(0) = 0$. $\int_0^1 u(t) dA(t)$ denotes the Riemann-Stieltjes integral of u with respect to A . k is a constant. Problems involving Riemann-Stieltjes integral boundary condition have been studied in [3, 7–9, 13]. These boundary conditions includes multipoint and integral boundary conditions, and sums of these, in a single framework. By changing variables $t \mapsto 1 - t$, studying (1.1) also covers the case

$$u(0) = \int_0^1 u(t) dA(t), \quad u(1) = 0.$$

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For a comprehensive study of the case when there is a Riemann-Stieltjes integral boundary condition at both ends, see [7].

In recent years, there are many papers investigating nonlocal boundary value problems of the second order ordinary differential equation $u'' + f(t, u) = 0$. For example, we refer the reader to [1, 3–5, 7–9, 11, 12] for some work on problems with integral type boundary conditions. However, there are fewer papers investigating boundary value problems of the equation $-u'' + k^2 u = f(t, u)$. In [6], Du and Zhao investigated the following multi-point boundary value problem

$$\begin{cases} -u'' = f(t, u), & t \in (0, 1), \\ u(0) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), & u(1) = 0. \end{cases}$$

They assumed f is decreasing in u and get existence of $C[0, 1]$ positive solutions ω with the property that $\omega(t) \geq m(1-t)$ for some $m > 0$. In a recent paper [5], Webb and Zima studied the problem (1.1) (and others) when dA is allowed to be a signed measure, and obtained existence of multiple positive solutions under suitable conditions on $f(t, u)$. Here we only study the positive measure case. We impose stronger restrictions on f . We suppose f is increasing in u , satisfies a strong sublinear property and may be singular at $t = 0, 1$. By applying the monotone iterative technique, we obtain the existence and uniqueness of $C^1[0, 1]$ positive solutions in some set D . Also, we use iterative methods, we establish uniqueness, obtain error estimates and the convergence rate of $C^1[0, 1]$ positive solutions with the property that there exists $M > m > 0$ such that $mt \leq u(t) \leq Mt$.

In this paper, we first introduce some preliminaries and lemmas in Section 2, and then we state our main results in Section 3.

2 Preliminaries and lemmas

We make the following assumptions:

(H_1) There exists $k > 0$ such that $\sinh(k) > \int_0^1 \sinh(k(1-t))dA(t)$;

(H_2) $f \in C((0, 1) \times [0, +\infty), [0, +\infty))$, $f(t, u)$ is increasing in u and there exists a constant $b \in (0, 1)$ such that

$$f(t, ru) \geq r^b f(t, u), \quad \text{for all } r \in (0, 1) \text{ and } (t, u) \in (0, 1) \times [0, +\infty). \quad (1.2)$$

Remark 2.1. If $M > 1$, condition (1.2) is equivalent to

$$f(t, Mu) \leq M^b f(t, u), \quad \text{for all } (t, u) \in (0, 1) \times [0, +\infty). \quad (1.3)$$

Our discussion is in the space $E = C[0, 1]$ of continuous functions endowed with the usual supremum norm. Let $P = \{u \in C[0, 1] : u \geq 0\}$ be the standard cone of nonnegative continuous functions.

Definition 2.1. A function $u \in C[0, 1] \cap C^2(0, 1)$ is called a $C[0, 1]$ solution if it satisfies (1.1). A $C[0, 1]$ solution u is called a $C^1[0, 1]$ solution if both $u'(0+)$ and $u'(1-)$ exist. A solution u is called a positive solution if $u(t) > 0$, $t \in (0, 1)$.

The Green's function for (1.1) is given in the following Lemma which was proved in [5] for the general case when dA is a signed measure.

Lemma 2.1 [5] Suppose that $g \in C(0, 1)$ and (H_1) holds. Then the following linear boundary value problem

$$\begin{cases} -u'' + k^2 u = g(t), & t \in (0, 1), \\ u(0) = 0, & u(1) = \int_0^1 u(t) dA(t) \end{cases} \quad (2.1)$$

has a unique positive solution u and u can be expressed in the form

$$u(t) = \int_0^1 F(t, s) g(s) ds,$$

where

$$F(t, s) = G(t, s) + \frac{\sinh(kt)}{\sinh(k) - \int_0^1 \sinh(k\tau) dA(\tau)} \int_0^1 G(\tau, s) dA(\tau), \quad s, t \in [0, 1], \quad (2.2)$$

$$G(t, s) = \begin{cases} \frac{\sinh(ks) \sinh(k(1-t))}{\sinh(k) k}, & 0 \leq s \leq t, \\ \frac{\sinh(kt) \sinh(k(1-s))}{\sinh(k) k}, & t \leq s \leq 1. \end{cases} \quad (2.3)$$

Remark 2.2. We call $F(t, s)$ the Green's function of problem (1.1). Suppose that (H_1) , (H_2) hold. Then solutions of (1.1) are equivalent to continuous solutions of the integral equation

$$u(t) = \int_0^1 F(t, s) f(s, u(s)) ds,$$

where $F(t, s)$ is mentioned in (2.2).

Lemma 2.2 For any $t, s \in [0, 1]$, there exist constants $c_1, c_2 > 0$ such that

$$c_2 e(t) e(s) \leq F(t, s) \leq c_1 e(s), \quad s, t \in [0, 1], \quad (2.4)$$

where $e(s) = s(1-s)$.

Proof. Suppose that

$$I(t) = \sinh(k)t - \sinh(kt), \quad t \in [0, 1].$$

Then $I(0) = I(1) = 0$ and $I''(t) = -k^2 \sinh(kt) \leq 0$, $t \in [0, 1]$. So $I(t) \geq 0$, i.e.

$$\sinh(kt) \leq \sinh(k)t, \quad t \in [0, 1]. \quad (2.5)$$

Similarly we have

$$kt \leq \sinh(kt), \quad t \in [0, 1]. \quad (2.6)$$

From (2.3) we know

$$\frac{k}{\sinh(k)} G(t, t) G(s, s) \leq G(t, s) \leq G(t, t). \quad (2.7)$$

By using (2.3), (2.5) and (2.6) we obtain

$$G(t, t) \geq \frac{(kt)(k(1-t))}{\sinh(k)k} = \frac{ke(t)}{\sinh(k)}, \quad (2.8)$$

and

$$G(t, t) \leq \frac{(\sinh(k)t)(\sinh(k)(1-t))}{\sinh(k)k} = \frac{\sinh(k)e(t)}{k}. \quad (2.9)$$

From (2.2), (2.7), (2.8) and (2.9) we have

$$F(t, s) \geq G(t, s) \geq \frac{k}{\sinh(k)} G(t, t) G(s, s) \geq \left(\frac{k}{\sinh(k)}\right)^3 e(t)e(s) \quad (2.10)$$

and

$$\begin{aligned} F(t, s) &\leq G(s, s) + G(s, s) \frac{\sinh(k)}{\sinh(k) - \int_0^1 \sinh(k\tau) dA(\tau)} \int_0^1 dA(\tau) \\ &\leq \frac{\sinh(k)}{k} e(s) \left[1 + \frac{\sinh(k)}{\sinh(k) - \int_0^1 \sinh(k\tau) dA(\tau)} \int_0^1 dA(\tau) \right]. \end{aligned} \quad (2.11)$$

Letting $c_1 = \frac{\sinh(k)}{k} \left[1 + \frac{\sinh(k)}{\sinh(k) - \int_0^1 \sinh(k\tau) dA(\tau)} \int_0^1 dA(\tau) \right]$ and $c_2 = \left(\frac{k}{\sinh(k)}\right)^3$, we have $c_2 e(t)e(s) \leq F(t, s) \leq c_1 e(s)$.

Thus, (2.4) holds.

3 Main results

Now we state the main results as follows.

Theorem 3.1 Suppose that (H_1) , (H_2) hold. Let $D = \{u(t) \in C[0, 1] \mid \exists L_u \geq l_u > 0, l_u t \leq u(t) \leq L_u t, t \in [0, 1]\}$. If

$$0 < \int_0^1 f(t, t) dt < +\infty \quad (3.1)$$

holds. Then problem (1.1) has a unique $C^1[0, 1]$ positive solution u^* in D . Moreover, for any initial $x_0 \in D$, the sequence of functions defined by

$$x_n = \int_0^1 F(t, s) f(s, x_{n-1}(s)) ds, \quad n = 1, 2, \dots$$

converges uniformly to the unique solution $u^*(t)$ on $[0, 1]$ as $n \rightarrow \infty$. Furthermore, we have the error estimation

$$\|x_n(t) - u^*(t)\| \leq 2(1 - (t_0^2)^{b^n})\|v_0\|, \quad (3.2)$$

where t_0, v_0 are defined below, and $F(t, s)$ is mentioned in (2.2).

Proof. From $u(t) \in D$ we know there exists $L_u > 1 > l_u > 0$ such that

$$l_u s \leq u(s) \leq L_u s, \quad s \in [0, 1].$$

This, together with (H_2) , (1.2) and (1.3), implies that

$$(l_u)^b f(s, s) \leq f(s, u(s)) \leq f(s, L_u s) \leq (L_u)^b f(s, s), \quad s \in (0, 1). \quad (3.3)$$

Let us define an operator T by

$$Tu = \int_0^1 F(t, s) f(s, u(s)) ds, \quad u \in D. \quad (3.4)$$

From (3.1) and (3.3) and Lemma 2.2 we can have

$$\int_0^1 F(t, s) f(s, u(s)) ds \leq c_1 (L_u)^b \int_0^1 s(1-s) f(s, s) ds < +\infty.$$

So the integral operator T makes sense. By (2.2), (2.3), (2.5), (2.6) and (2.7), we have that

$$\begin{aligned} F(t, s) &\geq \sinh(kt) \frac{\int_0^1 G(\tau, s) dA(\tau)}{\sinh(k) - \int_0^1 \sinh(k\tau) dA(\tau)} \\ &\geq kt \frac{\int_0^1 G(\tau, s) dA(\tau)}{\sinh(k) - \int_0^1 \sinh(k\tau) dA(\tau)}, \end{aligned} \quad (3.5)$$

$$\begin{aligned} F(t, s) &\leq G(t, t) + \frac{\sinh(kt)}{\sinh(k) - \int_0^1 \sinh(k\tau) dA(\tau)} \int_0^1 G(\tau, s) dA(\tau) \\ &= \sinh(kt) \left(\frac{\sinh(k(1-t))}{\sinh(k)k} + \frac{\int_0^1 G(\tau, s) dA(\tau)}{\sinh(k) - \int_0^1 \sinh(k\tau) dA(\tau)} \right) \\ &\leq t \sinh(k) \left(\frac{1}{k} + \frac{\int_0^1 G(\tau, s) dA(\tau)}{\sinh(k) - \int_0^1 \sinh(k\tau) dA(\tau)} \right). \end{aligned} \quad (3.6)$$

Thus

$$Tu(t) \geq t \frac{k(l_u)^b \int_0^1 \left(\int_0^1 G(\tau, s) f(s, s) ds \right) dA(\tau)}{\sinh(k) - \int_0^1 \sinh(k\tau) dA(\tau)}, \quad t \in [0, 1], \quad (3.7)$$

$$Tu(t) \leq t(L_u)^b \sinh(k) \times \int_0^1 \left(\frac{1}{k} + \frac{\int_0^1 G(\tau, s) dA(\tau)}{\sinh(k) - \int_0^1 \sinh(k\tau) dA(\tau)} \right) f(s, s) ds, \quad t \in [0, 1]. \quad (3.8)$$

Thus, from (3.1), (3.7) and (3.8), we obtain

$$T : D \rightarrow D.$$

It is known from Remark 2.2 that a fixed point of the operator T is a solution of BVP (1.1).

From condition (1.2) we obtain

$$T(ru) = \int_0^1 F(t, s) f(s, ru(s)) ds \geq r^b \int_0^1 F(t, s) f(s, u(s)) ds = r^b Tu, \quad (3.9)$$

Obviously T is an increasing operator and from (1.3) we have

$$T(Mu) \leq M^b Tu. \quad (3.10)$$

Let $x_0 \in D$ be given. Choose $t_0 \in (0, 1)$ such that

$$t_0^{1-b} x_0 \leq Tx_0 \leq \left(\frac{1}{t_0}\right)^{1-b} x_0.$$

Let us define $u_0 = t_0 x_0$, $v_0 = \frac{1}{t_0} x_0$, $t_0 \in (0, 1)$. Then $u_0 \leq v_0$ and from (3.9) and (3.10) we have

$$Tu_0 \geq t_0^b Tx_0 \geq t_0 x_0 = u_0, \quad Tv_0 \leq \left(\frac{1}{t_0}\right)^b Tx_0 \leq \frac{1}{t_0} x_0 = v_0. \quad (3.11)$$

Now we define

$$u_n = Tu_{n-1}, \quad v_n = Tv_{n-1}, \quad (n = 1, 2, 3, \dots).$$

It is easy to verify from (3.11) that

$$u_0 \leq u_1 \leq \dots \leq u_n \leq \dots \leq v_n \leq \dots \leq v_1 \leq v_0. \quad (3.12)$$

Clearly, $u_0 = t_0^2 v_0$. By induction, we see that

$$u_n \geq (t_0^2)^{b^n} v_n, \quad (n = 0, 1, 2, \dots). \quad (3.13)$$

Since P is a normal cone with normality constant 1, it follows that

$$\|v_n - u_n\| \leq \|u_{n+p} - u_n\| \leq (1 - (t_0^2)^{b^n})\|v_0\|. \quad (3.14)$$

So $\{u_n\}$ is a cauchy sequence, therefore u_n converges to some $u^* \in D$. From this inequality it also follows that $v_n \rightarrow u^*$.

We see that u^* is a fixed point of T . Thus, $u^* \in D$ from $u_0, v_0 \in D$ and $u^* \in [u_0, v_0]$. It follows from $u_0 \leq x_0 \leq v_0$ that $u_n \leq x_n \leq v_n$, ($n = 1, 2, 3, \dots$). So

$$\begin{aligned} \|x_n - u^*\| &\leq \|x_n - u_n\| + \|u_n - u^*\| \leq 2\|v_n - u_n\| \\ &\leq 2(1 - (t_0^2)^{b^n})\|v_0\|. \end{aligned} \quad (3.15)$$

Next we prove the uniqueness of fixed points of T . Let $\bar{x} \in D$ be any fixed points of T . From u^* , $\bar{x} \in D$ and the definition of D , we can put $t_1 = \sup\{t > 0 \mid \bar{x} \geq tu^*\}$. Evidently $0 < t_1 < \infty$. We now prove $t_1 \geq 1$. In fact, if $0 < t_1 < 1$, then

$$\bar{x} = T\bar{x} \geq T(t_1 u^*) \geq (t_1)^b T u^* = (t_1)^b u^*,$$

which contradicts the definition of t_1 since $(t_1)^b > t_1$. Thus $t_1 \geq 1$ and $\bar{x} \geq u^*$. In the same way, we can prove $\bar{x} \leq u^*$ and hence $\bar{x} = u^*$. The uniqueness of fixed points of A in D is proved. For any initial $z_0 \in D$, $z_n = T^n z_0 \rightarrow u^*$ with rate of convergence

$$\|z_n - u^*\| = o(1 - (t_0^2)^{b^n}) \quad (3.16)$$

from the results above. Choosing $z_0 = x_0$, we obtain

$$\|x_n - u^*\| = o(1 - (t_0^2)^{b^n}). \quad (3.17)$$

This completes the proof of Theorem 3.1.

Remark Suppose that $\beta_i(t)$ ($i = 0, 1, 2, \dots, m$) are nonnegative continuous functions on $(0, 1)$, which may be unbounded at the end points of $(0, 1)$. Ω is the set of functions $f(t, u)$ which satisfy the condition (H_2) . Then we have the following conclusions:

- (1) $\beta_i(t) \in \Omega$, $u^b \in \Omega$, where $0 < b < 1$;
- (2) If $0 < b_i < +\infty$ ($i = 1, 2, \dots, m$) and $b > \max_{1 \leq i \leq m} \{b_i\}$, then $[\beta_0(t) + \sum_{i=1}^m \beta_i(t) u^{b_i}]^{\frac{1}{b}} \in \Omega$;
- (3) If $f(t, u) \in \Omega$, then $\beta_i(t) f(t, u) \in \Omega$;
- (4) If $f_i(t, u) \in \Omega$ ($i = 1, 2, \dots, m$), then $\max_{1 \leq i \leq m} \{f_i(t, u)\} \in \Omega$, $\min_{1 \leq i \leq m} \{f_i(t, u)\} \in \Omega$.

The above four facts can be verified directly. This indicates that there are many kinds of functions which satisfy the condition (H_2) .

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